

TENSOR PRODUCT ACTIONS WITH THE TRACIAL ROKHLIN PROPERTY ON UHF ALGEBRAS

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ABSTRACT. We show that for any countable discrete maximally almost periodic group G and any UHF algebra A , there is a strongly outer action α of G on A that preserves some tensor product decomposition of A . When G is also elementary amenable, α enjoys the tracial Rokhlin property and the crossed product $A \rtimes_{\alpha} G$ is of tracial rank zero, has a unique tracial state, and satisfies the Universal Coefficient Theorem. We then contrast this to existence of actions with the (pointwise) Rokhlin property.

INTRODUCTION

Some of the earliest and most fundamental examples of unital simple C^* -algebras are the so-called uniformly hyperfinite (UHF) algebras, which were studied and classified by Dixmier [3] and Glimm [6]. Despite the classical status of UHF algebras in the theory of C^* -algebras, many new insights can still be gained from studying them and their group actions. One way to define a UHF algebra is to start with a sequence $(n_k)_{k \in \mathbb{N}}$ of strictly positive integers and then associate to it the C^* -algebra $M_{(n_k)_{k \in \mathbb{N}}}$ using an infinite tensor product (see Definition 1.10). That is

$$M_{(n_l)_{l \in \mathbb{N}}} = M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_l} \otimes \cdots$$

We immediately see that such algebras are unital separable simple and possess a unique tracial state. Glimm showed that such algebras are completely determined up to isomorphism by the prime decomposition of $\prod_{l=1}^{\infty} n_l$ when considered as a *generalised integer*. That is by the numbers (with values in $\mathbb{N} \cup \{\infty\}$)

$$N_p(n_l)_{l \in \mathbb{N}} = \sup\{r \mid p^r \text{ divides } n_1 n_2 \cdots n_m \text{ for some } m\}$$

for all prime numbers p . This data can be summarised in a more sophisticated way that was generalised by Elliott to determine all of the simple approximately finite dimensional C^* -algebras up to isomorphism, namely as an “ordered group of projections” or the ordered K_0 -group, which for UHF algebras is the ordered subgroup of \mathbb{R} generated by

$$\{n_1^{-1} n_2^{-1} \cdots n_m^{-1} \mid m \in \mathbb{N}\}.$$

The success of this initiated the *Elliott program* for classification of simple C^* -algebras that has had far reaching consequences. The statement of the problem is sometimes referred to as the *Elliott Conjecture*.

Looking at these UHF algebras together with their tensor product decomposition can lead us to another interesting possibility. That when the numbers $N_p((n_l)_{l \in \mathbb{N}})$ lie in $\{0, \infty\}$ for all primes p , the algebra $M_{(n_l)_{l \in \mathbb{N}}}$ is *strongly self-absorbing* in the sense of Winter [25]. This perspective has had considerable success and has allowed one to *localise the Elliott conjecture* at a UHF algebra (see Winter [26]). Lin, Niu [18] and Winter [26] made use of this to extend Lin's celebrated classification of unital simple separable nuclear C^* -algebras A of tracial rank at most one and satisfying the Universal Coefficient Theorem (UCT) to those A that only had to have tracial rank at most one *after* tensoring by a strongly self-absorbing UHF algebra.

With these developments in mind we study group actions on UHF algebras. Given our definition, It is natural to look at those group actions that preserve some tensor product decomposition. Furthermore we will for convenience look at those actions that are *inner* on each factor. So that for each $l \in \mathbb{N}$, we have a group homomorphism

$$G \rightarrow U(M_{n_l}) \rightarrow \text{Aut } M_{n_l}.$$

Putting this together we have (c.f Definition 1.11)

$$G \rightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \rightarrow \text{Aut} \left(\bigotimes_{l=1}^{\infty} M_{n_l} \right) = \text{Aut } M_{(n_l)_{l \in \mathbb{N}}}.$$

These represent the class of actions that are most accessible to study. Among these actions we look for examples of actions satisfying some sort of *Rokhlin property*, *tracial Rokhlin property* or *strong outerness*, listed from strongest to weakest. The actions with these properties represent the unknown that we are trying to investigate and hope to gain insight from. We do this to obtain model actions for a property that is promised to produce crossed product C^* -algebras that have tracial rank zero and satisfy the UCT, which belongs to a class that satisfies the Elliott Conjecture (Lin [17]). Part of the novelty of this is that we now have at our disposal all of the groups that have been incorporated recently into Matui-Sato's definition of the tracial Rokhlin property ([19]), which at least includes the countable discrete elementary amenable groups. The previous investigations have focused on finite groups ([4], [5], [10], [8]) or single automorphisms of infinite order ([9], [1], [14]), which can be considered as actions by the group of integers, and later to higher rank free abelian groups ([22], [21]). The Klein bottle group was the first example of an infinite non-abelian group and its

actions were classified by Matui-Sato [19]. We will give a model tensor product action for this group in Section 4. These investigations, like in Matui-Sato [19], usually prove a *uniqueness* result for the actions, which will be lacking in this investigation. We focus purely on proving *existence* and our search can be considered an extension of the theme started in [24] where it was shown that actions with the tracial Rokhlin property essentially always exist. Now we see for UHF algebras if they can always exist as a tensor product action.

If the action is outer, then the group homomorphism must be injective. In particular, we have an embedding

$$G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}).$$

These groups are known as *maximally almost periodic* groups, so our question, in its weakest version, becomes

Question. *Which maximally almost periodic groups can have strongly outer tensor product actions on a UHF algebra?*

We show in Section 2 that the answer is essentially “all of them”. That is we arrive at Theorem 2.6, which says

Theorem. *If G is a countable discrete maximally almost periodic group and A is any UHF algebra, then there is a strongly outer tensor product action of G on A . If G is also elementary amenable then α has the tracial Rokhlin property.*

We then give some examples of strongly outer abelian group actions in Section 3. In Section 4 we compare and contrast the abundance of strongly outer tensor product actions with the scarcity of actions with the Rokhlin property. We avoid a proper definition of the Rokhlin property for G by using a substitute called the *pointwise Rokhlin property* that serves to illustrate our point. In this case we quickly realise that the Universal UHF algebra \mathcal{Q} maximises our chances of finding group actions and we arrive at Theorem 4.5.

Theorem. *If G is a countable discrete almost abelian group, then there is a tensor product action of G on \mathcal{Q} with the (pointwise) Rokhlin property.*

In Section 5 we emphasise the convenience of the properties we have imposed when investigating the crossed products formed. For example, we know that being an inner action on each tensor factor will guarantee us that the crossed product is amenable quasidiagonal and satisfies the Universal Coefficient Theorem (UCT). While for example, strong

outererness tells us that the crossed product will have a unique tracial state, among other things.

Theorem. *Suppose G is a countable discrete maximally almost periodic amenable group, A is any UHF algebra and α is a tensor product action of G on A with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ is unital simple separable nuclear with tracial rank zero, satisfies the Universal Coefficient Theorem and has a unique tracial state. Furthermore, if G is almost abelian, then $A \rtimes_\alpha G$ is also locally type I.*

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Notation. We adopt the following conventions throughout this article:

- The letter G is reserved for groups, usually countable and discrete.
- The letter A is reserved for C^* -algebras, usually unital separable and simple.
- Write $a \approx_\epsilon b$ to stand for $\|a - b\| < \epsilon$.
- Denote the cardinality of a set S by $|S|$.

1. PRELIMINARIES

1.1. Discrete groups. Here we introduce the main classes of groups that we will be dealing with.

Definition 1.1 ((F, ϵ) -invariance). Let G be a countable discrete group, let $F \subset G$ be a finite subset and let $\epsilon > 0$. We say a finite subset K of G is (F, ϵ) -invariant if $K \neq \emptyset$ and

$$\left| K \cap \bigcap_{g \in F} g^{-1}K \right| \geq (1 - \epsilon)|K|.$$

Definition 1.2 (Elementary amenable). A countable discrete group G is said to be *amenable* if an (F, ϵ) -invariant subset exists from any (F, ϵ) . The group G is said to be *elementary amenable* if it is contained in the smallest class of groups that contains all abelian groups, all finite groups and is closed under taking subgroups, quotients, direct limits and extensions.

Definition 1.3 (Maximally almost periodic). A discrete group G is said to be maximally almost periodic if for every $g \in G \setminus \{1\}$ there is a finite dimensional unitary representation of G where g acts non-trivially. Equivalently, G is *maximally almost periodic* if there is an embedding

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n).$$

Lemma 1.4. *If G is a discrete maximally almost periodic group, then there is an embedding*

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n)$$

such that the image of G intersects $\prod_{n=1}^{\infty} \mathbb{C}1_n$ trivially.

Proof. We give a proof most conveniently explained for countable discrete groups. It suffices to show that there is an embedding

$$G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l})$$

for some sequence $(n_l)_{l \in \mathbb{N}}$ such that the image of G intersects $\prod_{l=1}^{\infty} \mathbb{C}1_{l(g)}$ trivially. Now starting with any embedding

$$\varphi : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n)$$

we write φ_n for the map $G \rightarrow U(M_n)$. For any $g \in G$ not 1, there is an $l(g) \in \mathbb{N}$ such that $\varphi_{l(g)}(g) \neq 1$. If $\varphi_{l(g)}(g) = \lambda 1_{l(g)}$ for some scalar $\lambda \neq 1$, then we can replace $\varphi_{l(g)}$ by $\varphi_{l(g)}[g] : G \rightarrow U(M_{l(g)+1})$ defined by $h \mapsto \text{diag}(1, \varphi_{l(g)}(h))$ for all $h \in G$. Then we have for each $g \in G$ not 1 a map

$$\varphi_l[g] : G \rightarrow U(M_{l(g)+1})$$

such that $\varphi_lg \notin \mathbb{C}1_{l(g)}$. Combining these maps we get a map into the product

$$\varphi[G] : G \rightarrow \prod_{g \in G} U(M_{l(g)+1})$$

such that the image of G intersects $\prod_{l=1}^{\infty} \mathbb{C}1_{n_l}$ trivially. \square

Definition 1.5 (Almost abelian). A discrete group is said to be *almost abelian* if it has an abelian subgroup of finite index.

We will also observe throughout the course of this article that countable discrete almost abelian groups are maximally almost periodic.

Lemma 1.6. *Suppose that G is a discrete group with an abelian subgroup H of finite index. Then the subgroup*

$$N = \bigcap_{g \in G} gHg^{-1}$$

is normal and abelian with finite index. In particular, every almost abelian group is elementary amenable.

Proof. The subgroup N is abelian because it is a subgroup of H . It is a normal subgroup of G by construction. It suffices to show that G/N is finite in which case G is an extension of an abelian group N by a finite group G/N and we are done. Now N is exactly the kernel of the action of G on G/H , which is a finite set and hence G/N is isomorphic to a subgroup of the symmetric group $S_{[G:H]}$ making it finite. \square

Lemma 1.7. *Suppose G is a discrete group, H is a subgroup of G with finite index k and let*

$$N = \bigcap_{g \in G} gHg^{-1}.$$

If there is a unitary representation of H on \mathbb{C}^n with corresponding homomorphism

$$\rho_H : H \rightarrow U(M_n),$$

then there exists an induced representation with corresponding homomorphism given by

$$\rho_H^G : G \rightarrow U(M_n \otimes M_k)$$

such that the restriction of ρ_H^G to N is unitarily equivalent to k copies of the restriction of ρ_H to N . That is to say $\rho_H^G|_N = \rho_H|_N \otimes \text{id}_{M_k}$.

Proof. Let $\{g_1, \dots, g_k\}$ be a set of coset representatives for H and define the induced representation on \mathbb{C}^{nk} by first writing

$$\mathbb{C}^{nk} = g_1\mathbb{C}^n \oplus g_2\mathbb{C}^n \oplus \dots \oplus g_k\mathbb{C}^n,$$

which now suggests how the G action is defined. Let $v_1, \dots, v_l \in \mathbb{C}^n$ and let σ be the permutation of $\{1, \dots, k\}$ induced by G acting on G/H . We have that for each $j \leq k$ there exists $h_j \in H$ such that $gg_j = g_{\sigma(j)}h_j$. Then define

$$g(g_j v_j)_{j \leq k} = (g_{\sigma(j)}(h_j v_j))_{j \leq k}.$$

We can check that this defines a unitary representation of G with corresponding homomorphism

$$\rho_H^G : G \rightarrow U(M_n \otimes M_k),$$

Now let $h \in N$, we see that for each $j \leq k$ there is a unique h_j such that $h = g_j h_j g_j^{-1}$. This means that N preserves the decomposition

$$\mathbb{C}^{nk} = g_1 \mathbb{C}^n \oplus g_2 \mathbb{C}^n \oplus \cdots \oplus g_k \mathbb{C}^n.$$

We also see that with the obvious basis, h has the following diagonal matrix representation:

$$h \mapsto \text{diag}(\rho_H|_N(h_1), \rho_H|_N(h_2), \dots, \rho_H|_N(h_k)).$$

This map is unitarily equivalent to the embedding

$$\rho_H|_N \otimes 1_{M_k} : N \rightarrow U(M_n) \otimes 1_{M_k} \hookrightarrow U(M_n) \otimes U(M_k),$$

with matrix representation

$$h \mapsto \rho_H|_N(h) \otimes 1_{M_k},$$

via the unitary matrix $\oplus_{j \leq k} (\rho_H^G(g_j)^*|_{g_j \mathbb{C}^n})$. \square

Lemma 1.8. *Suppose G is a discrete almost abelian group. Then every irreducible representation of G is finite dimensional. Furthermore, there exists an $M \in \mathbb{N}$ such that the dimension of every irreducible representation is at most M .*

Proof. There being a uniform bound on the irreducible representations is due to Kaplansky [13, Theorem 1 and Theorem 3]. \square

Proposition 1.9. *Every countable discrete abelian group is isomorphic to a subgroup of*

$$\left(\bigoplus_{n=1}^{\infty} \mathbb{Q} \right) \oplus \left(\bigoplus_{n=1}^{\infty} \mathbb{Q}/\mathbb{Z} \right).$$

Proof. We sketch a proof relying on the basic theory of abelian groups (see for example, Kaplansky [12]). First every abelian group is a subgroup of a divisible group. Every divisible group is a direct sum of copies of \mathbb{Q} and $\mathbb{Q}[1/p] \subset \mathbb{Q}/\mathbb{Z}$. So then every abelian group is a subgroup of a direct sum of \mathbb{Q} s and \mathbb{Q}/\mathbb{Z} s. When our groups are countable we will need at most countably many summands. \square

1.2. Tensor product actions on UHF algebras. We will introduce the universal UHF algebra \mathcal{Q} and other UHF algebras as infinite tensor products of full matrix algebras.

Definition 1.10. Consider the direct limit of C^* -algebras given by

$$\varinjlim (M_{n!}, \phi_n : a \mapsto \text{diag}(a, \dots, a)).$$

This can be represented as an infinite tensor product of matrices by first writing

$$M_{n!} = M_1 \otimes \cdots \otimes M_n$$

so that the limit and its connecting maps can be reexpressed and we define

$$\mathcal{Q} = \varinjlim \left(\bigotimes_{i=1}^n M_i, (\bigotimes_{i=1}^n \text{id}_i) \otimes 1_{n+1} \right).$$

This limit gives meaning to the expression for \mathcal{Q} as

$$\bigotimes_{n=1}^{\infty} M_n = M_1 \otimes M_2 \otimes M_3 \otimes \dots$$

where we will sometimes identify $M_{n!}$ with its image under this decomposition,

$$M_{n!} = M_1 \otimes \dots \otimes M_n \otimes 1_{n+1} \otimes 1_{n+2} \otimes \dots$$

In a similar way we can define for any sequence of strictly positive integers $(n_l)_{l \in \mathbb{N}}$, its associated UHF algebra $M_{(n_l)_{l \in \mathbb{N}}}$ as

$$\bigotimes_{l=1}^{\infty} M_{n_l} = M_{n_1} \otimes M_{n_2} \otimes \dots \otimes M_{n_l} \otimes \dots$$

So we have in this notation

$$\mathcal{Q} = M_{(n)_{n \in \mathbb{N}}} = \bigotimes_{n=1}^{\infty} M_n.$$

Let $(n_l)_{l \in \mathbb{N}}$ and $(l_l)_{l \in \mathbb{N}}$ be sequences of strictly positive integers. We say that $(n_l)_{l \in \mathbb{N}}$ and $(m_l)_{l \in \mathbb{N}}$ are of the same *type* if

$$M_{(n_l)_{l \in \mathbb{N}}} \cong M_{(m_l)_{l \in \mathbb{N}}}.$$

We say $(n_l)_{l \in \mathbb{N}}$ is of *infinite type* if

$$M_{(n_l)_{l \in \mathbb{N}}} \otimes M_{(n_l)_{l \in \mathbb{N}}} \cong M_{(n_l)_{l \in \mathbb{N}}}.$$

A special case of sequences of infinite type is the constant sequence $(n)_{l \in \mathbb{N}}$ for some $n \in \mathbb{N} \setminus \{0\}$, where we will sometimes adopt the notation

$$M_{n^\infty} = M_{(n_l)_{l \in \mathbb{N}}}$$

Since we have defined \mathcal{Q} and other UHF algebras as tensor products, it will be natural to look at the actions that preserve this definition. These can be represented by group homomorphisms that have the following factorisation:

$$G \rightarrow \prod_{l=1}^{\infty} \text{Aut } M_{n_l} \rightarrow \text{Aut} \left(\bigotimes_{l=1}^{\infty} M_{n_l} \right) = \text{Aut } M_{(n_l)_{l \in \mathbb{N}}}.$$

The group $\text{Aut } M_{n_l}$ is isomorphic to the quotient of $U(M_n)$, by its center $\mathbb{C}1_n$. We will actually impose a stronger condition for what we will

consider to be *tensor product actions*, that the action on each factor M_{n_l} be *inner* as well.

Definition 1.11 (Tensor product action). Define for any sequence of strictly positive integers $(n_l)_{l \in \mathbb{N}}$ the group homomorphism

$$\text{Ad}(n_l)_{l \in \mathbb{N}} : \prod_{l=1}^{\infty} U(M_{n_l}) \rightarrow \text{Aut } M_{(n_l)_{l \in \mathbb{N}}},$$

written Ad when there is no confusion, for $u_{n_l} \in U(M_{n_l})$ by

$$(u_{n_l})_{l=1}^{\infty} \mapsto \bigotimes_{l=1}^{\infty} \text{Ad } u_{n_l}$$

on the algebraic direct limit and then take the extension to the C^* -direct limit. We consider an action of G to be a *tensor product action* if it is represented by a group homomorphism that has the following factorisation

$$G \rightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \xrightarrow{\text{Ad}} \text{Aut } M_{(n_l)_{l \in \mathbb{N}}}$$

for some strictly positive sequence $(n_l)_{l \in \mathbb{N}}$.

We will now try to make explicit some ways to manipulate our definitions. For example, we may regroup and permute the factors in a tensor product decomposition and the algebras will remain of the same type via a canonical isomorphism and any tensor product actions will remain tensor product actions after conjugating by this canonical isomorphism.

Lemma 1.12 (Regrouping). *Let $(l_i)_{i=1}^{\infty}$ be a strictly increasing sequence with $l_1 = 0$ and $l_2 > 1$, then define m_i by*

$$m_i = \prod_{l_i+1}^{l_{i+1}} n_j.$$

In this case there is a canonical isomorphism

$$\Psi : M_{(n_l)_{l \in \mathbb{N}}} \rightarrow M_{(m_i)_{i \in \mathbb{N}}}$$

with

$$\Psi(M_{n_{l_i+1}} \otimes \cdots \otimes M_{n_{l_{i+1}}}) = M_{m_i}$$

and a canonical embedding

$$\varphi : \prod_{l=1}^{\infty} U(M_{n_l}) \rightarrow \prod_{i=1}^{\infty} U(M_{m_i})$$

such that the following diagram commutes:

$$\begin{array}{ccc} \prod_{l=1}^{\infty} U(M_{n_l}) & \xrightarrow{\text{Ad}} & \text{Aut } M_{(n_l)_{l \in \mathbb{N}}} \\ \varphi \downarrow & & \downarrow \Psi(\cdot) \Psi^{-1} \\ \prod_{i=1}^{\infty} U(M_{m_i}) & \xrightarrow{\text{Ad}} & \text{Aut } M_{(m_i)_{i \in \mathbb{N}}} \end{array}$$

Proof. In terms of the direct limit definition of the infinite tensor product this corresponds to taking a subsequence of the connecting maps. Hence this will give the same limit with the new groupings preserved by the action. \square

Lemma 1.13 (Reordering). *Let $d : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ be any function and let $\sigma : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be any bijection. Then there is a $*$ -isomorphism*

$$\Psi_{\sigma} : \bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}$$

such that

$$\Psi_{\sigma}(M_{d(m,n)}) = M_{d(\sigma^{-1}(m,n))}$$

and a canonical isomorphism

$$\varphi_{\sigma} : \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_{d(m,n)}) \rightarrow \prod_{l=1}^{\infty} U(M_{d(\sigma(l))})$$

given by

$$((u_{d(m,n)})_{n=1}^{\infty})_{m=1}^{\infty} \mapsto (u_{\sigma(\sigma^{-1}(m,n))})_{\sigma^{-1}(m,n)=1}^{\infty}$$

such that the following diagram commutes

$$\begin{array}{ccc} \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_{d(m,n)}) & \xrightarrow{\otimes \text{Ad}} & \text{Aut} \left(\bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \right) \\ \varphi_{\sigma} \downarrow & & \downarrow \Psi_{\sigma}(\cdot) \Psi_{\sigma}^{-1} \\ \prod_{l=1}^{\infty} U(M_{d(\sigma(l))}) & \xrightarrow{\text{Ad}} & \text{Aut} \left(\bigotimes_{l=1}^{\infty} M_{d(\sigma(l))} \right) \end{array}$$

Proof. We first note that the domain algebra is simple so any unital $*$ -homomorphism we define will be injective. Now we proceed to define the map. Let $m, n \in \mathbb{N}$ and let $\Psi_{m,n} : M_{d(m,n)} \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}$ be defined on matrix units $e_{i,j}^{(d(m,n))} \in M_{d(m,n)}$ by

$$e_{i,j}^{(d(m,n))} \mapsto \left(\bigotimes_{l=1}^{\sigma^{-1}(m,n)} 1_{d(\sigma(l))} \right) \otimes e_{i,j}^{(d(m,n))} \otimes \left(\bigotimes_{l=\sigma^{-1}(m,n)+1}^{\infty} 1_{d(\sigma(l))} \right).$$

for $1 \leq i, j \leq d(m,n)$. We note for later that for $m, n \in \mathbb{N}$

$$\Psi_{m,n}(M_{d(m,n)}) = M_{d(\sigma(\sigma^{-1}(m,n)))}.$$

Let $N > 1$. We see that since σ is injective, the images of $\Psi_{n,m}$ are in different tensor factors and in particular commute for $n = 1, \dots, N$. Therefore we get a map

$$\bigotimes_{n=1}^N \Psi_{m,n} : \bigotimes_{n=1}^N M_{d(m,n)} \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}$$

satisfying for $e^{(d(m,n))} \in M_{d(m,n)}$ that

$$\bigotimes_{n=1}^N e^{(d(m,n))} \mapsto \prod_{n=1}^N \Psi_{m,n}(e^{(d(m,n))}).$$

Restricting this to $(\bigotimes_{n=1}^{N-1} M_{d(m,n)}) \otimes 1_{d(m,N)}$ we get

$$\left(\bigotimes_{n=1}^{N-1} e^{(d(m,n))} \right) \otimes 1_{d(m,N-1)} \mapsto \left(\prod_{n=1}^N \Psi_{m,n}(e^{(d(m,n))}) \right) \Psi_{m,N}(1_{d(m,N)}).$$

which agrees with $\bigotimes_{n=1}^{N-1} \Psi_{m,n}$. Hence there is a map

$$\Psi_m : \varinjlim_N \left(\bigotimes_{n=1}^N M_{d(m,n)}, \text{id} \otimes 1_{d(m,n)} \right) \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}.$$

We see that the images commute for $m = 1, \dots, M$ since σ is injective so we have a map

$$\bigotimes_{m=1}^M \Psi_m : \bigotimes_{m=1}^M \bigotimes_{n=1}^{\infty} M_{d(m,n)} \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))},$$

defined for $x_m \in \bigotimes_{n=1}^{\infty} M_{d(m,n)}$ by

$$\bigotimes_{m=1}^M x_m \mapsto \prod_{m=1}^M \Psi_m(x_m).$$

Restricting this to $(\bigotimes_{m=1}^{M-1} \bigotimes_{n=1}^{\infty} M_{d(m,n)}) \otimes 1$ we get

$$\bigotimes_{m=1}^{M-1} x_m \otimes 1 \mapsto \left(\prod_{m=1}^{M-1} \Psi_m(x_m) \right) \Psi_M(1)$$

which agrees with the map $\bigotimes_{m=1}^{M-1} \Psi_m$. Hence we have a map

$$\Psi : \varinjlim_M \bigotimes_{m=1}^M \bigotimes_{n=1}^{\infty} M_{d(m,n)} \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))},$$

that is,

$$\Psi_\sigma : \bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \rightarrow \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}.$$

Since Ψ_σ restricts to $\Psi_{m,n}$ on $M_{d(m,n)}$ we have that

$$\Psi_\sigma(M_{d(m,n)}) = M_{d(\sigma(\sigma^{-1}(m,n)))},$$

which also implies that surjectivity, since these generate the target algebra as m and n run through \mathbb{N} by surjectivity of σ . Hence Ψ_σ is a $*$ -isomorphism. We also see that the diagram commutes. \square

1.3. Crossed products by tensor product actions. For any unital C^* -algebra A , any discrete group G and an action α of G on A , define the crossed product $A \rtimes_\alpha G$ as the C^* -algebra with the presentation

$$A \rtimes_\alpha G = \langle a, u_g \mid a \in A, g \in G, \alpha_g(a) = u_g a u_g^* \rangle.$$

For $g \in G$ we will refer to u_g as the canonical unitary implementing α_g . We see that for any unital C^* -algebra B whenever there is a map unital embedding $\Psi : A \rightarrow B$ and a group homomorphism $\varphi : G \rightarrow U(B)$ such that for all $a \in A$ and all $g \in G$,

$$\varphi(g)\Psi(a)\varphi(g)^* = \Psi(\alpha_g(a)),$$

then there is a canonical $*$ -homomorphism $A \rtimes_\alpha G \rightarrow B$.

We give a direct limit decomposition of $M_{(n_l)_{l \in \mathbb{N}}} \rtimes_\alpha G$ when α is a tensor product action with corresponding homomorphism of the form

$$\alpha : G \rightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \xrightarrow{\text{Ad}} \text{Aut } M_{(n_l)_{l \in \mathbb{N}}},$$

Let α_m denote the restriction of α to $M_{(n_l)_{l \leq m}}$ and $\alpha_{g,m}$ denote the restriction of α_g . Let 1_m be the identity in M_{n_m} and let

$$\phi_m : M_{(n_l)_{l \leq m}} \rightarrow M_{(n_l)_{l \leq m+1}}$$

be defined by $x \mapsto x \otimes 1_m$. Let

$$\phi_{n,\infty} : M_{(n_l)_{l \leq m}} \rightarrow M_{(n_l)_{l \in \mathbb{N}}}$$

be the canonical embedding into the direct limit. For all $g \in G$, write $u_g^{(m)}$ for the canonical unitary implementing $\alpha_{g,m}$ in $M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G$.

We have for the group homomorphism defined by $g \mapsto u_g^{(m+1)}$ that

$$\begin{aligned} u_g^{(m+1)} \phi_m(x) (u_g^{(m+1)})^* &= (\alpha_g)_{m+1}(x \otimes 1_{m+1}) \\ &= \alpha_{g,m}(x) \otimes 1_{m+1} \\ &= \phi_m(\alpha_{g,m}(x)). \end{aligned}$$

Hence there is a $*$ -homomorphism

$$\psi_m : M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G \rightarrow M_{(n_l)_{l \leq m+1}} \rtimes_{\alpha_{m+1}} G$$

on generators $x \in M_{(n_l)_{l \leq m}}$ and $u_g^{(m)}$ for $g \in G$ by

$$xu_g^{(m)} \mapsto \phi_m(x)u_g^{(m+1)}.$$

Denote by $\psi_{m,\infty}$ the canonical map to $\varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m)$ given by

$$\frac{\{(x_m) \in \prod_m M_{(n_l)_{l \leq m}} \mid \exists n : \psi_k(x_k) = x_{k+1} \forall k > n\}}{\{(x_m) \mid \lim_m x_m = 0\}}$$

we see that for $x \in M_{(n_l)_{l \leq m}}$ and $g \in G$ by

$$\psi_{m,\infty}(xu_g^{(m)}) = (0, \dots, 0, xu_g^{(m)}, \phi_m(x)u_g^{(m+1)}, (\phi_{m+1} \circ \phi_m)(x)u_g^{(m+2)}, \dots).$$

Lemma 1.14. *Let $\alpha : G \rightarrow \text{Aut } M_{(n_l)_{l \in \mathbb{N}}}$ be a tensor product action preserving the natural decomposition of $M_{(n_l)_{l \in \mathbb{N}}}$. Then,*

$$M_{(n_l)_{l \in \mathbb{N}}} \rtimes_{\alpha} G \cong \varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m).$$

Proof. We will define $*$ -homomorphisms in both directions and check they are inverse to each other. Let u_g implement α_g in $M_{(n_l)_{l \in \mathbb{N}}} \rtimes_{\alpha} G$. Similarly to the case for ψ_m we can define a map

$$\psi'_{m,\infty} : M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G \rightarrow M_{(n_l)_{l \in \mathbb{N}}} \rtimes_{\alpha} G$$

on generators $x \in M_{(n_l)_{l \leq m}}$ and $u_g^{(n)}$ for $g \in G$ by

$$xu_g^{(m)} \mapsto \phi_{m,\infty}(x)u_g.$$

We check that

$$\begin{aligned} (\psi'_{m+1,\infty} \circ \psi_m)(xu_g^{(m)}) &= \psi'_{m+1,\infty}(\phi_m(x)u_g^{(m+1)}) \\ &= \phi_{m+1,\infty}(\phi_m(x))u_g \\ &= \phi_{m,\infty}(x)u_g \\ &= \psi'_{m,\infty}(xu_g^{(m)}). \end{aligned}$$

So we have a map

$$\varinjlim (\psi'_{m,\infty}) : \varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m) \rightarrow M_{(n_l)_{l \in \mathbb{N}}} \rtimes_{\alpha} G.$$

Now we define the inverse map. We have a map

$$\psi_{m,\infty}|_{M_{(n_l)_{l \leq m}}} : M_{(n_l)_{l \leq m}} \rightarrow \varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m)$$

and we check for $x \in M_{(n_l)_{l \leq m}}$

$$\begin{aligned} (\psi_{m+1,\infty} \circ \phi_m)(x) &= (0, \dots, 0, \phi_m(x), (\phi_{m+1} \circ \phi_m)(x), \dots) \\ &= (0, \dots, 0, x, \phi_m(x), (\phi_{m+1} \circ \phi_m)(x), \dots) \\ &= \psi_{m,\infty}(x). \end{aligned}$$

So we have a map from the universal property of the limit

$$\phi : M_{(n_l)_{l \in \mathbb{N}}} \rightarrow \varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m).$$

now define a group homomorphism

$$G \rightarrow U(\varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m))$$

for $g \in G$ by

$$g \mapsto (u_g^{(m)})_{m \in \mathbb{N}}$$

and check the covariance condition on the generators. Let $x \in M_{(n_l)_{l \leq m}}$ and $g \in G$. We have

$$\begin{aligned} (u_g^{(m)})_{n \in \mathbb{N}} \phi(x) (u_g^{(m)})_{m \in \mathbb{N}}^* &= (0, \dots, 0, u_g^{(m)} x (u_g^{(m)})^*, u_g^{(m+1)} \phi_m(x) (u_g^{(m+1)})^*, \dots) \\ &= (0, \dots, 0, \alpha_m(x), \alpha_{m+1}(\phi_m(x)), \dots) \\ &= (0, \dots, 0, \alpha_m(x), \phi_m(\alpha_m(x)), \dots) \\ &= \phi(\alpha_m(x)) \\ &= \phi(\alpha(x)). \end{aligned}$$

Hence there is a map

$$\psi : M_{(n_l)_{l \in \mathbb{N}}} \rtimes_{\alpha} G \rightarrow \varinjlim (M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G, \psi_m),$$

which we now check is inverse to $\varinjlim (\psi'_{m,\infty})$ on generators. Let $x \in M_{(n_l)_{l \leq m}}$ and let $g \in G$. We have

$$\begin{aligned} (\psi \circ \varinjlim (\psi'_{m,\infty}))(\psi_{m,\infty}(x u_g^{(m)})) &= \psi(\psi'_{m,\infty}(x u_g^{(m)})) \\ &= \psi(\phi_{m,\infty}(x) u_g) \\ &= \psi_{m,\infty}(x u_g^{(m)}). \end{aligned}$$

Let $x \in M_{(n_l)_{l \leq m}}$ and let $g \in G$. We have

$$\begin{aligned} (\varinjlim (\psi'_{m,\infty}) \circ \psi)(\phi_{m,\infty}(x) u_g) &= \varinjlim (\psi'_{m,\infty})(\phi(\phi_{m,\infty}(x))(u_g^{(m)})_{m \in \mathbb{N}}) \\ &= \varinjlim (\psi'_{m,\infty})(\psi_{m,\infty}(x)(u_g^{(m)})_{m \in \mathbb{N}}) \\ &= \psi'_{m,\infty}(x u_g^{(m)}) \\ &= \phi_{m,\infty}(x) u_g. \end{aligned}$$

□

Define the (full) group C^* -algebra $C^*(G)$ of a group G as the C^* -algebra with the presentation

$$C^*(G) = \langle g \in G \mid g^* = g^{-1} \rangle.$$

Lemma 1.15. *If G is a countable discrete amenable maximally almost periodic group, then $C^*(G)$ is amenable quasidiagonal and satisfies the UCT. Furthermore, if G is almost abelian then $C^*(G)$ is type I.*

Proof. That $C^*(G)$ is amenable when G is amenable is due to Guichardet [7]. That $C^*(G)$ is quasidiagonal when G is amenable and maximally almost periodic is discussed in the introduction of [2] among other things. That $C^*(G)$ satisfies the UCT can be attributed to G satisfying the Baum-Connes conjecture under these assumptions. That $C^*(G)$ is type I when G is almost abelian follows from Lemma 1.8. \square

Lemma 1.16. *The map $\theta_m : M_{(n_l)_{l \leq m}} \otimes C^*(G) \rightarrow M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G$ defined by*

$$\theta_m : x \otimes g \mapsto x(g_1^{-1} \otimes \cdots \otimes g_n^{-1})u_g$$

is a $$ -isomorphism. With inverse defined by*

$$\theta_m^{-1} : xu_g \mapsto x(g_1 \otimes \cdots \otimes g_n) \otimes g.$$

Proof. Note that the action of $\text{Ad}((g_1^{-1} \otimes \cdots \otimes g_n^{-1})u_g)$ on $M_{(n_l)_{l \leq m}}$ is by design trivial and hence $(g_1^{-1} \otimes \cdots \otimes g_n^{-1})u_g$, which is the image of $1 \otimes g$, commutes with the image of $x \otimes 1$, which lies in $M_{(n_l)_{l \leq m}}$. Therefore θ_m is well-defined. For θ_m^{-1} , we have that

$$\begin{aligned} \theta_m^{-1}(u_g)x &= ((g_1 \otimes \cdots \otimes g_n) \otimes g)(x \otimes 1) \\ &= ((g_1 \otimes \cdots \otimes g_n)x) \otimes g \\ &= \alpha_g(x)\theta_m^{-1}(u_g), \end{aligned}$$

and hence θ_m^{-1} is well-defined. Therefore both maps are well-defined by the universal properties of the presentations for these algebras. The fact that they are inverses only needs to be checked on the generators where it is obvious, hence θ_m and θ_m^{-1} are $*$ -isomorphisms. \square

We arrive at the following theorem.

Theorem 1.17. *Let A be a UHF algebra, let G be a discrete group and let α be a tensor product action of G on A preserving the decomposition $A \cong M_{(n_l)_{l \in \mathbb{N}}}$ for some sequence $(n_l)_{l \in \mathbb{N}}$. Let g_m be the image of g in $U(M_{n_m})$ and suppose*

$$\Phi_m : M_{(n_l)_{l \leq m}} \otimes C^*(G) \rightarrow M_{(n_l)_{l \leq m}} \otimes C^*(G)$$

is a $$ -homomorphism defined on generators by*

$$x \otimes g \mapsto ((x \otimes 1_{n_{m+1}})g_{n+1}) \otimes g.$$

Then

$$A \rtimes_{\alpha} G \cong \varinjlim (\Phi_m, M_{(n_l)_{l \leq m}} \otimes C^*(G)).$$

Proof. From Lemmas 1.14 and 1.16, it suffices to define the maps Φ_m so that the following diagrams commute

$$\begin{array}{ccc} M_{(n_l)_{l \leq m}} \rtimes_{\alpha_m} G & \xrightarrow{\psi_m} & M_{(n_l)_{l \leq m+1}} \rtimes_{\alpha_m} G \\ \theta_m \uparrow & & \downarrow \theta_{m+1}^{-1} \\ M_{(n_l)_{l \leq m}} \otimes C^*(G) & \xrightarrow{\Phi_m} & M_{(n_l)_{l \leq m+1}} \otimes C^*(G). \end{array}$$

We have for $x \in M_{(n_l)_{l \leq m}}$ and $g \in G$ that

$$\begin{aligned} \Phi_m(x \otimes g) &= (\theta_{m+1}^{-1} \circ \psi_m \circ \theta_m)(x \otimes g) \\ &= (\theta_{m+1}^{-1} \circ \psi_m)(x(g_1^{-1} \otimes \cdots \otimes g_m^{-1})u_g^{(m)}) \\ &= \theta_{m+1}^{-1}(\phi_m(x(g_1^{-1} \otimes \cdots \otimes g_m^{-1}))u_g^{(m+1)}) \\ &= \phi_m(x(g_1^{-1} \otimes \cdots \otimes g_m^{-1}))((g_1 \otimes \cdots \otimes g_{m+1}) \otimes g) \\ &= (x(g_1^{-1} \otimes \cdots \otimes g_m^{-1}) \otimes 1_{m+1})((g_1 \otimes \cdots \otimes g_{m+1}) \otimes g) \\ &= ((x \otimes 1_{m+1})g_{m+1}) \otimes g \\ &= (\phi_m(x)g_{m+1}) \otimes g. \end{aligned}$$

□

Corollary 1.18. *Suppose G is a countable discrete amenable maximally almost periodic group and suppose A is any UHF algebra. If α is a tensor product action of G on A , then $A \rtimes_{\alpha} G$ is nuclear quasidiagonal and satisfies the UCT. Furthermore, if G is almost abelian then $A \rtimes_{\alpha} G$ is locally type I.*

1.4. The tracial Rokhlin property. Here we recall the definitions of the tracial Rokhlin property for countable discrete amenable groups given by Matui-Sato [19]. We specialise it to the case of UHF algebras.

Definition 1.19 (Matui-Sato tracial Rokhlin property). A group action α of G on a C^* -algebra A has the *tracial Rokhlin property* if for every finite subset $F \subset G$ and $\epsilon > 0$, there is a finite (F, ϵ) -invariant subset K in G and a central sequence $(p_n)_{n \in \mathbb{N}}$ in A consisting of projections such that for $g, h \in K$ with $g \neq h$

- $\lim_{n \rightarrow \infty} \alpha_g(p_n)\alpha_h(p_n) = 0$
- $\lim_{n \rightarrow \infty} \max_{\tau \in T(A)} |\tau(p_n) - |K|^{-1}| = 0.$

Since we will assume A is a UHF-algebra, which has a unique trace τ we can simplify the last condition. Also since UHF algebras are separable we can replace central sequences by requiring approximate commutativity with finite subsets of A . By functional calculus we can

replace approximately mutual orthogonality by exact mutually orthogonality at the cost of demoting transitivity to approximate transitivity. That is we have

Lemma 1.20. *A group action α of G on a UHF algebra A has the tracial Rokhlin property if and only if for every finite subset $F \subset G$, any $\epsilon > 0$, every finite subset $\{a_1, \dots, a_n\} \subset A$, there is a finite (F, ϵ) -invariant subset K in G , and mutually orthogonal projections $(p_k)_{k \in K}$ such that for all $h \in K$ and $g \in K \cap h^{-1}K$, and writing $p = \sum_{k \in K} p_k$, we have*

- $[p_h, a_i] \approx_\epsilon 0$ for all $i \leq n$,
- $\alpha_h(p_g) \approx_\epsilon p_{hg}$,
- $\tau(p) \approx_\epsilon 1$.

We now look at what the tracial Rokhlin property and Rokhlin property means for a single automorphism. This can be considered the $G = \mathbb{Z}$ and $G = \mathbb{Z}/n\mathbb{Z}$ cases of the above definition.

Definition 1.21. Let A be a unital C^* -algebra and $\alpha \in \text{Aut } A$. If α has infinite order, we say α has the tracial Rokhlin property, if for every $\epsilon > 0$, every finite subset $\{a_1, \dots, a_n\}$ in A , every $n' \in \mathbb{N}$, there exists $N' > n'$ mutually orthogonal projections $p_1, p_2, \dots, p_{N'}$ such that

- $[p_i, a_j] \approx_\epsilon 0$ for all $i \leq N'$ and all $j \leq n$,
- $\alpha(p_i) \approx_\epsilon p_{i+1}$ for all $i \leq N' - 1$,
- $\tau(p) \approx_\epsilon 1$.

If this is true with $p = 1$, we say that α has the Rokhlin property. If α has finite order k , then we ignore n' , require $N' = k$ and $\alpha(p_k) \approx_\epsilon p_1$. Again, if $p = 1$ we say that α has the Rokhlin property. Clearly then the Rokhlin property is stronger than the tracial Rokhlin property.

Definition 1.22. If a group G α has for every $g \in G$ not 1 that α_g has the Rokhlin property, then α is said to have the *pointwise Rokhlin property*. If instead α_g has the tracial Rokhlin property for all $g \in G$ not 1, then α is said to have the *pointwise tracial Rokhlin property*.

Proposition 1.23. *Define a tensor product action of \mathbb{Z} on M_{2^∞} via the homomorphisms*

$$\mathbb{Z} \rightarrow U(M_{2^l})$$

such that for a fixed primitive 2^l -th root of unity ξ we have

$$1 \mapsto \text{diag}(1, \xi, \dots, \xi^{2^l-1}).$$

Let α be the action given by the homomorphisms above as follows

$$\alpha : \mathbb{Z} \hookrightarrow \prod_{l=1} U(M_{2^l}) \rightarrow \text{Aut } M_{(2^l)_{l \in \mathbb{N}}}.$$

Then α has the pointwise Rokhlin property.

Proof. Since the image of k will just be α_1^k this will only permute the diagonal entries so it suffices to show that α_1 has the Rokhlin property. Let $\epsilon > 0$, let a_1, \dots, a_n be a finite subset of M_{2^∞} and let $n' \in \mathbb{N}$. Let $l > n'$ be large enough that there exists $a'_i \in \bigotimes_{j=1}^l M_{2^j}$ for $i \leq n$ such that $a_i \approx_{\epsilon/2} a'_i$. We find 2^{l+1} projections in $(\bigotimes_{j=1}^l M_{2^j}) \otimes M_{2^{l+1}}$ which α_1 cycles. We see that by definition α_1 is equivalent to a cyclic permutation matrix of order 2^{l+1} when acting on $M_{2^{l+1}}$. So there are matrix units $e_{i,j}$ such that $\alpha_1(e_{i,j}) = e_{i+1,j+1}$ with indices modulo 2^{l+1} . Then let $p_i = e_{i,i}$. We check that

- $[p_i, a_j] \approx_\epsilon [p_i, a'_i] = 0$ for all $i \leq N'$ and all $j \leq n$,
- $\alpha_1(p_i) \approx_\epsilon p_{i+1}$ for all $i \leq 2^{l+1} - 1$,
- $\sum_i p_i = \sum_i e_{i,i} = 1$.

□

Proposition 1.24. *Let G be a finite group and define an action of G on $\mathbb{C}^{|G|}$ via its left regular representation. This gives a group homomorphism*

$$G \rightarrow U(M_{|G|}),$$

which we duplicate infinitely many times to get

$$\alpha : G \hookrightarrow \prod_{n=1} U(M_{|G|}) \rightarrow \text{Aut } M_{|G|^\infty}.$$

Then α has the (pointwise) Rokhlin property.

Proof. We will show that α has the pointwise Rokhlin property. Let $g \in G$ with $g \neq 1$ and let $\langle g \rangle$ denote the subgroup generated by g . If we restrict the left regular representation of G to $\langle g \rangle$, we get a decomposition into a direct sum of copies of the left regular representation for $\langle g \rangle$. Hence it suffices to assume that $G = \langle g \rangle$ and that $|G| = n$, the order of g . We will now show that there are projections p_1, \dots, p_n in M_n satisfying

- $\alpha_g(p_i) = p_{i+1}$ for $i \leq n - 1$ and $\alpha_g(p_n) = p_1$.
- $\sum_{i=1}^n p_i = 1$.

For this write

$$\mathbb{C}^n = \mathbb{C}1 \oplus \mathbb{C}g \oplus \cdots \oplus \mathbb{C}g^{n-1},$$

so that g acts via the unitary defined by

$$g^i \mapsto g^{i+1}$$

and let p_i be the projection onto $\mathbb{C}g^{i-1}$. We will automatically satisfy the approximate commutativity condition because every finite subset is approximately contained in finitely many tensor factors. \square

Lemma 1.25. *Let A and B be unital nuclear C^* -algebras, let $\alpha \in \text{Aut } A$ and let $\beta \in \text{Aut } B$. If α has the same order as $\alpha \otimes \beta$ and satisfies the Rokhlin property (resp. tracial Rokhlin property), then $\alpha \otimes \beta$ satisfies the Rokhlin property (resp. tracial Rokhlin property). If we only know that α^k has the Rokhlin property (resp. tracial Rokhlin property) for some $k > 1$ but we also know that β has finite order k and has the Rokhlin property (resp. tracial Rokhlin property), then $\alpha \otimes \beta$ has the Rokhlin property (resp. tracial Rokhlin property).*

Proof. Assume first that α has infinite order, whence so does $\alpha \otimes \beta$. Let $\epsilon > 0$, let $\{x_1, \dots, x_n\}$ be a finite subset of $A \otimes B$ and let $n' \in \mathbb{N}$. Since the algebraic tensor product is dense in $A \otimes B$, there exists $N \in \mathbb{N}$, $a_{i,j} \in A$ and $b_{i,j} \in B$ for all $i \leq N$ and $j \leq n$ such that

$$x_j \approx_\epsilon \sum_{i=1}^N a_{i,j} \otimes b_{i,j} \text{ for } 1 \leq j \leq n.$$

Let

$$\delta = \frac{\epsilon}{\max_{j \leq n} (\sum_{i=1}^N \|b_{i,j}\|, \sum_{i=1}^N \|a_{i,j}\|)}.$$

Since α^k has the tracial Rokhlin property, there exists $N' > n'$ and mutually orthogonal projections $e_l \in A$ for $l \leq N'$ such that if we write $e = e_1 + e_2 + \dots + e_{N'}$, then we have

- $[e_l, a_{i,j}] \approx_\delta 0$ for all $i \leq N$ and $j \leq n$,
- $[e_l, \alpha^{1-m}(a_{i,j})] \approx_\delta 0$ for all $i \leq N$, all $j \leq n$ and all $m \leq k$,
- $\alpha^k(e_l) \approx_\epsilon e_{l+1}$ for all $1 \leq l \leq N' - 1$,
- $\tau_A(e) \approx_\epsilon 1$ for all $\tau_A \in T(A)$ ($e = 1$ if α^k has the Rokhlin property).

If we know α has the tracial Rokhlin property, then set $p_l = e_l \otimes 1$ and $p = \sum_{l=1}^{N'} p_l$. Otherwise suppose β has finite order k and the tracial Rokhlin property. Then there exists mutually orthogonal projections

$f_m \in A$ for $m \leq k$ such that if we set $f = f_1 + f_2 + \dots + f_k$ and $f_{k+1} = f_1$, then we have

- $[f_m, b_{i,j}] \approx_\delta 0$ for all $i \leq N$ and $j \leq n$,
- $\beta(f_m) \approx_\epsilon f_{m+1}$ for all $1 \leq m \leq k$,
- $\tau_B(f) \approx_\epsilon 1$ for all $\tau_B \in T(B)$ ($f = 1$ if β has the Rokhlin property).

In this case, set $p_{l,m} = \alpha^{m-1}(e_l) \otimes f_m$ for all $l \leq N$ and $1 \leq m \leq k$ ordered lexicographically, set $p = \sum_{l,m} p_{l,m}$. We now check that these witness the tracial Rokhlin property for $\alpha \otimes \beta$. Note that we can think of the first case as $k = 1$, $f_1 = 1_B$ and $p_{l,1} = p_l$ of the second case if we disregard β so that we can verify both cases simultaneously.

We first check that our projections approximately commute with our given finite subset. Let $1 \leq j \leq n$ and let $1 \leq m \leq k$. We have

$$\begin{aligned}
p_{l,m} x_j &= (\alpha^{m-1}(e_l) \otimes f_m) x_j \\
&\approx_\epsilon (\alpha^{m-1}(e_l) \otimes f_m) \left(\sum_{i=1}^N a_{i,j} \otimes b_{i,j} \right) \\
&= \sum_{i=1}^N \alpha^{m-1}(e_l) a_{i,j} \otimes f_m b_{i,j} \\
&= \sum_{i=1}^N a_{i,j} \alpha^{m-1}(e_l) \otimes f_m b_{i,j} + \sum_{i=1}^N [\alpha^{m-1}(e_l), a_{i,j}] \otimes f_m b_{i,j} \\
&\approx_\epsilon \sum_{i=1}^N a_{i,j} \alpha^{m-1}(e_l) \otimes f_m b_{i,j} \\
&= \sum_{i=1}^N a_{i,j} \alpha^{m-1}(e_l) \otimes b_{i,j} f_m + \sum_{i=1}^N a_{i,j} \alpha^{m-1}(e_l) \otimes [f_m, b_{i,j}] \\
&\approx_\epsilon \sum_{i=1}^N a_{i,j} \alpha^{m-1}(e_l) \otimes b_{i,j} f_m \\
&= \left(\sum_{i=1}^N a_{i,j} \otimes b_{i,j} \right) (\alpha^{m-1}(e_l) \otimes f_m) \\
&\approx_\epsilon x_j (\alpha^{m-1}(e_l) \otimes f_m) \\
&= x_j p_{l,m}.
\end{aligned}$$

We now check that β cycles the projections. Let $1 \leq l \leq N' - 1$ and $m \leq k$. We have

$$\begin{aligned} (\alpha \otimes \beta)(p_{l,m}) &= (\alpha \otimes \beta)(\alpha^{m-1}(e_l) \otimes f_m) \\ &= \alpha^{m+1}(e_l) \otimes f_{m+1} \\ &\approx_\epsilon p_{l,m+1} \quad \text{if } m \neq k. \end{aligned}$$

If $m = k$, we see from the same calculation as before

$$\begin{aligned} (\alpha \otimes \beta)(p_{l,m}) &\approx_\epsilon \alpha^k(e_l) \otimes f_{k+1} \\ &\approx_\epsilon e_{l+1} \otimes f_1 \\ &= p_{l+1,1}. \end{aligned}$$

The remaining condition on the trace is seen to be satisfied as follows:

$$\begin{aligned} \tau(p) &= \tau\left(\sum_{l,m} p_{l,m}\right) \\ &= \tau(e \otimes f) \\ &= \tau_A(e) \otimes \tau_B(f) \\ &\approx_{2\epsilon} 1. \end{aligned}$$

($p = e \otimes f = 1$ if we are talking about the Rokhlin property.) Letting $k = 1$, $f_1 = 1_B$ and $p_{l,1} = p_l$ in the above calculations will give us the required calculations for p_l .

When α has finite order but α^k still has the (tracial) Rokhlin property, the same argument can be used with N' replaced by the order of α^k and no mention of n' . \square

1.5. Strong outerness.

Definition 1.26 (Weakly inner and strongly outer). Let A be a UHF algebra and let τ be the unique tracial state. Write (π_τ, H_τ) for the Gelfand-Naimark-Segal (GNS) representation for τ . Then $\alpha \in \text{Aut } A$ is weakly inner if when extended to an automorphism of $\pi_\tau(A)''$, the weak closure of the $\pi_\tau(A)$ in $B(H_\tau)$, there is a unitary $u \in \pi_\tau(A)''$ such that $\alpha = \text{Ad } u$. An action is called strongly outer if α_g is not weakly inner for all $g \in G \setminus \{1\}$.

In the case an automorphism α is weakly inner we have for any τ

$$\pi_\tau(\alpha(a)) = u\pi_\tau(a)u^*.$$

This means that there is a representation of the crossed product

$$\begin{aligned} \pi : A \rtimes \mathbb{Z} &\rightarrow B(H_\tau) \\ \pi|_A &= \pi_\tau \\ \pi(u_1) &= u. \end{aligned}$$

Definition 1.27. Let $\widetilde{1}_A \in H_\tau$ be the vector obtained from 1_A in the GNS construction. Then we have the following positive state on $A \rtimes \mathbb{Z}$ extending τ ;

$$\tau_{A \rtimes \mathbb{Z}}(x) = \langle \pi(x) \widetilde{1}_A, \widetilde{1}_A \rangle$$

Lemma 1.28. $\tau_{A \rtimes \mathbb{Z}}$ is a trace on $A \rtimes \mathbb{Z}$.

Proof. By linearity and continuity of $\tau_{A \rtimes \mathbb{Z}}$ it suffices to show

$$\tau_{A \rtimes \mathbb{Z}}(au^m bu^n) = \tau_{A \rtimes \mathbb{Z}}(bu^n au^m)$$

for all $m, n \in \mathbb{Z}$, $a, b \in A$. We first show

$$\tau_{A \rtimes \mathbb{Z}}(au^m bu^n) = \tau_{A \rtimes \mathbb{Z}}(bu^n au^m)$$

for $m, n \geq 0$ by induction on $l = m + n$. This is clear for $l = 0$ since $\tau_{A \rtimes \mathbb{Z}}|_A = \tau_A$ is a trace.

Now let $v_k \rightarrow u$ weakly with $v_k \in A$. For the diagram below, the top equality is the induction hypothesis, the vertical arrows represent convergence, while the equality in the bottom row is the conclusion drawn from these facts, completing the induction step.

$$\begin{array}{ccc} \tau(av_k u^{m-1} bu^n) & \stackrel{=}{=} & \tau(bu^k av_k u^{m-1}) \\ \downarrow & & \downarrow \\ \tau(au^m bu^n) & \stackrel{=}{=} & \tau(bu^n au^m) \end{array}$$

Now we complete the proof in the case $m < 0$ or $n < 0$, by replacing the corresponding u by u^* in the argument above since v_k^* converges weakly to u^* . \square

Define for $a \in A$, $\|a\|_2 = \tau(a^*a)^{1/2}$. The next lemma gives us a sufficient condition for strong outerity. It was essentially taken from Matui-Sato [19] and we would like to take the opportunity to thank Y. Sato for communicating a detailed proof of their lemma.

Lemma 1.29. Let A_n be a sequence of unital simple nuclear C^* -algebras with unique tracial state and let $\alpha_n \in \text{Aut}(A_n)$. For $A = \bigotimes_{n=1}^\infty A_n$ define $\alpha \in \text{Aut } A$ by

$$\alpha = \bigotimes_{n=1}^\infty \alpha_n.$$

If there is a sequence of unitaries $v_n \in U(A_n)$ such that $\|\alpha_n(v_n) - v_n\|_2$ does not converge to 0, then α is not weakly inner.

Proof. Define a (central) sequence

$$v(n) = 1 \otimes \cdots \otimes 1 \otimes v_n \otimes 1 \otimes \cdots,$$

where 1 appears in every factor except A_n . We will show that if α is weakly inner then

$$\|\alpha(v(n)) - v(n)\|_2 \rightarrow 0,$$

Now assume there is a unitary $u \in \pi_\tau(A)''$ such that $\alpha_g^{\otimes \mathbb{N}} = \text{Ad } u$ on $\pi_\tau(A)''$. Then there is a sequence $(x_k)_{k \in \mathbb{N}}$ in $\pi_\tau(A)$ such that $x_k \rightarrow u$. This assumption gives an action of $A \rtimes_\alpha \mathbb{Z}$ on the trace representation of A as described in the previous section.. Let $\epsilon > 0$, fix k so that $\|u - x_k\|_{2, A \rtimes \mathbb{Z}} \approx_{\epsilon/2} 0$ by way of x_k strongly converging to u , and let n be large enough so that $[x_k, v(n)] \approx_\epsilon 0$, which is possible because $v(n)$ is a central sequence. We now calculate:

$$\begin{aligned} \|\alpha_g^{\otimes \mathbb{N}}(v(n)) - v(n)\|_{2, A} &= \|uv(n)u^* - v(n)\|_{2, A \rtimes \mathbb{Z}} \\ &= \|uv(n) - v(n)u\|_{2, A \rtimes \mathbb{Z}} \\ (\|ab\|_2 \leq \|a\|_2 \|b\|) &\leq 2\|u - x_k\|_{2, A \rtimes \mathbb{Z}} + \|x_k v(n) - v(n)x_k\| \\ &\approx_\epsilon \|x_k v(n) - v(n)x_k\| \\ &\approx_\epsilon 0. \end{aligned}$$

□

Theorem 1.30 (Matui-Sato). *Let A be a UHF algebra and let G be a countable discrete elementary amenable group. Then an action of G on A has the tracial Rokhlin property if and only if it is strongly outer. In particular, an automorphism has the tracial Rokhlin property if and only if it is not weakly inner.*

Proof. This is Matui-Sato [19, Theorem 3.7] specialized to actions of elementary amenable groups on UHF algebras. For automorphisms, let $G = \mathbb{Z}$ or $G = \mathbb{Z}/n\mathbb{Z}$. The case of single automorphisms was already known to Kishimoto and others. □

1.6. Crossed products by outer actions. We note here the advantages of the tracial Rokhlin property for determining the crossed product. The following is based on Kishimoto [16, Lemma 4.3].

Lemma 1.31. *Suppose α is an action of G on A and τ is any tracial state on $A \rtimes_\alpha G$. Then $\tau(au_g) = 0$ for all $a \in A$ and all $g \in G \setminus \{1\}$ such that α_g has the tracial Rokhlin property.*

Proof. Let $g \in G \setminus \{1\}$, let $a \in A$ and $\epsilon > 0$. Without loss of generality also assume $\|a\| \leq 1$. By definition of the tracial Rokhlin property, there exists mutually orthogonal projections p_1, \dots, p_n such that

$$\bullet [p_i, a] \approx_{\epsilon/2} 0$$

- $\alpha_g(p_i) \approx_{\epsilon/2} p_{i+1}$ for $i \leq n-1$
- $\tau(p) \approx_{\epsilon^2} 1$.

First we show that for $i \leq n-1$

$$(1) \quad \|p_i a \alpha(p_i)\| < \epsilon.$$

Let $i \leq n-1$. We have

$$\begin{aligned} p_i a \alpha(p_i) &\approx_{\epsilon/2} a p_i \alpha(p_i) \\ &\approx_{\epsilon/2} a p_i p_{i+1} \\ &= 0. \end{aligned}$$

Now we have

$$\begin{aligned} \tau(a u_g) &= \tau(p a u_g) + \tau((1-p) a u_g) \\ &\leq \tau(p a u_g) + \tau(1-p)^{1/2} \tau(a)^{1/2} \\ &\approx_{\epsilon} \tau(p a u_g) \\ &= \sum_{i=1}^n \tau(p_i a u_g) \\ &= \sum_{i=1}^n \tau(p_i a u_g p_i) \\ &= \sum_{i=1}^n \tau(p_i a \alpha_g(p_i) u_g p_i) \\ &\leq \sum_{i=1}^n \tau(p_i \|p_i a \alpha_g(p_i)\|^2 p_i)^{1/2} \tau(p_i)^{1/2} \\ &< \sum_{i=1}^n \epsilon \tau(p_i) \\ &= \epsilon \tau(p) \\ &\approx_{\epsilon^2} \epsilon \end{aligned}$$

Hence $\tau(a u_g) \approx_{2\epsilon+\epsilon^2} 0$. □

Proposition 1.32. *Let G be a countable discrete group, let A be a UHF algebra and let α be an action of G on A . Then we have*

- (i) *if G is amenable and α is outer, then $A \rtimes_{\alpha} G$ is nuclear and simple,*

- (ii) if α has the pointwise tracial Rokhlin property, then $A \rtimes_{\alpha} G$ has a unique tracial state,
- (iii) if G is amenable and α has the tracial Rokhlin property, then $A \rtimes_{\alpha} G$ is \mathcal{Z} -stable and has real rank zero.

Proof. We find general results from the literature and specialise them to our case.

- (i) Rosenberg [11] shows that crossed products are amenable when G is amenable. Simplicity follows from Kishimoto's [15, Theorem 3.1] if the action is outer.
- (ii) Lemma 1.31 shows that there is at most one trace τ on $A \rtimes_{\alpha} G$, which we check that τ defines a trace. Since finite sums of the form $\sum_g au_g$ are dense in $A \rtimes_{\alpha} G$ and τ is linear, it suffices to for all $a, b \in A$ and all $g, h \in G$ that

$$\tau(au_g bu_h) = \tau(bu_h au_g).$$

Let $a, b \in A$ and let $g, h \in G$. If $g \neq h$, then

$$\begin{aligned} \tau(au_g bu_h) &= \tau(a\alpha_g(b)u_{gh}) \\ &= 0 \\ &= \tau(b\alpha_h(a)u_{hg}) \\ &= \tau(bu_h au_g). \end{aligned}$$

Otherwise, we just need to show that

$$\tau(a\alpha_g(b)) = \tau(b\alpha_{g^{-1}}(a)),$$

which follows from τ restricting to a G -invariant trace on A . That is

$$\begin{aligned} \tau(a\alpha_g(b)) &= \tau(\alpha_g(\alpha_{g^{-1}}(a)b)) \\ &= \tau(\alpha_{g^{-1}}(a)b) \\ &= \tau(b\alpha_{g^{-1}}(a)). \end{aligned}$$

- (iii) Matui-Sato [19, Theorem 4.9] tells us the crossed product is \mathcal{Z} -stable. In the presence of \mathcal{Z} -stability we use a characterisation of real rank zero from Rordam[23, Corollary 7.3(ii)]. Since the image $K_0(A)$ under the trace is already dense in \mathbb{R} and the image of $K_0(A \rtimes_{\alpha} G)$ contains the image of $K_0(A)$, then $A \rtimes_{\alpha} G$ has real rank zero.

□

2. STRONGLY OUTER TENSOR PRODUCT ACTIONS

We will present here our results concerning strongly outer tensor product actions on UHF algebras. We essentially show that every group that can act with a tensor product action can also act with a strongly outer tensor product action. We will show this arbitrary UHF algebras, which will reduce to the case of the universal UHF algebra \mathcal{Q} (or any UHF algebra of infinite type). The assumption that G is an elementary amenable group will be added later to upgrade the status of strongly outer actions to those with the tracial Rokhlin property. Recalling Definition 1.11, we consider actions corresponding to group homomorphisms of the form

$$G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \xrightarrow{\text{Ad}} \text{Aut } M_{(n_l)_{l \in \mathbb{N}}}.$$

We first look at the case of the Universal UHF algebra \mathcal{Q} . Since we have chosen a standard tensor product decomposition for \mathcal{Q} to be $M_{(n)_{n \in \mathbb{N}}}$, we will look for tensor product action in the following standard form:

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q}.$$

There are other possible tensor product decompositions of \mathcal{Q} and we now show that no generality is lost from trying to find actions in standard form.

Lemma 2.1. *Let G be a countable discrete group and suppose that for some sequence $(n_l)_{l \in \mathbb{N}}$ in \mathbb{N} there exists a strongly outer action*

$$\alpha : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \xrightarrow{\text{Ad}} \text{Aut } M_{(n_l)_{l \in \mathbb{N}}}.$$

Then there exists a strongly outer action

$$\beta : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q}.$$

Proof. First we can if necessary regroup the tensor factors so that we can assume the sequence $(n_l)_{l \in \mathbb{N}}$ is strictly increasing. Let id denote the identity automorphism on $\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1} < n < n_l} M_n$ and consider the strongly outer action $\alpha \otimes \text{id}$

$$\alpha \otimes \text{id} : G \rightarrow \text{Aut} \left(\bigotimes_{l=1}^{\infty} M_{n_l} \right) \otimes \left(\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1} < n < n_l} M_n \right).$$

which factors through

$$\left(\prod_{l=1}^{\infty} U(M_{n_l}) \right) \times \left(\prod_{l=1}^{\infty} \prod_{n_{l-1} < n \leq n_l} U(M_n) \right) \xrightarrow{\text{Ad}}$$

Reordering the factors using Lemma 1.13, we get a conjugate action

$$\alpha \otimes \text{id} : G \rightarrow \text{Aut} \left(\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1} < n \leq n_l} M_n \right).$$

factoring through

$$\prod_{l=1}^{\infty} \prod_{n_{l-1} < n \leq n_l} U(M_n) \xrightarrow{\text{Ad}} \text{Aut} \left(\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1} < n \leq n_l} M_n \right).$$

Regrouping the factors now gives us our action in the required form. \square

So any groups that can act in another way can also be found to act in standard form on \mathcal{Q} . We now show that any discrete group can act on \mathcal{Q} via a strongly outer tensor product action in standard form, but first an easy lemma to help us verify strong outerity. The following lemma closely resembles and is taken from Matui-Sato [19, Lemma 6.13]. Thanks again to Y. Sato for communicating a proof to us.

Lemma 2.2. *Suppose A is a unital simple nuclear C^* algebra with a unique tracial state and $\alpha : G \rightarrow \text{Aut}(A)$ corresponds to an action of G on A and $\ker \alpha = \{1_G\}$. Then the action $\alpha^{\otimes \mathbb{N}}$ of G on $A^{\otimes \mathbb{N}} = \bigotimes_{n=1}^{\infty} A$ is strongly outer.*

Proof. We use Lemma 1.29 to show that for each $g \in G$ not 1, the automorphism that g acts by is not weakly inner. Since α is not trivial on A there is some $u \in A$ such that $\alpha(v) \neq v$. We can take v to be unitary since unitaries span A and α is linear. In particular, the sequence $\|\alpha_n(v_n) - v_n\|_2$ in Lemma 1.29 with $\alpha_n = \alpha$ and $v_n = v$ for all $n \in \mathbb{N}$ is constant and non-zero. \square

Proposition 2.3. *Suppose G is a discrete group with a tensor product action*

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q},$$

such that $\ker \alpha = \{\text{id}\}$. Then there exists a strongly outer tensor product action

$$\beta : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q}.$$

Proof. Consider the action $\alpha^{\otimes \mathbb{N}}$ of G on $\mathcal{Q}^{\otimes \mathbb{N}}$ defined for each $g \in G$, by

$$(\alpha^{\otimes \mathbb{N}})_g = \bigotimes_{n=1}^{\infty} \alpha_g.$$

We see that this action when restricted to G is strongly outer by Corollary 2.2 with $A = \mathcal{Q}$. We also see that $\alpha^{\otimes \mathbb{N}}$ factorises as

$$\alpha^{\otimes \mathbb{N}} : G \rightarrow \left(\prod_{n=1}^{\infty} U(M_n) \right)^{\mathbb{N}} \rightarrow \text{Aut}(\mathcal{Q}^{\otimes \mathbb{N}}),$$

which can be written in the more expanded form

$$\alpha^{\otimes \mathbb{N}} : G \rightarrow \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_n) \rightarrow \text{Aut} \left(\bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_n \right).$$

Rearrange the factors as in Lemma 1.13 to get a conjugate map (also call it $\alpha^{\otimes \mathbb{N}}$)

$$\alpha^{\otimes \mathbb{N}} : G \hookrightarrow \prod_{n=1}^{\infty} U(M_{n!}) \rightarrow \text{Aut} \left(\bigotimes_{n=1}^{\infty} M_{n!} \right).$$

(If one wants to be more explicit with the application of Lemma 1.13, we have that $d(m, n) = n$ and σ defined by taking the sequence $(1, 1)$, $(2, 1)$, $(1, 2)$, $(3, 1)$, $(2, 2)$, $(1, 3)$ and so on where the pairs (m, n) such that $m + n = 2$ are exhausted first and then $m + n = 3$ and so on. We also have $\bigotimes_{m+n=l} M_{d(m,n)} = M_{(l-1)!}$.) Now we can apply Lemma 2.1 to get a strongly outer map β in the standard form required

$$\beta : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q}.$$

□

Corollary 2.4. *Let G be a discrete maximally almost periodic group. There exists a strongly outer tensor product action*

$$\beta : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q}.$$

Proof. Lemma 1.4 gives us an embedding that we can compose with Ad to get a tensor product action α of the form

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \rightarrow \text{Aut } \mathcal{Q}$$

such that $\ker \alpha = \{1\}$. Now Proposition 2.3 applies. □

We now show that no generality is lost when only considering \mathcal{Q} as opposed to the other UHF algebras. The existence of strongly outer tensor product actions of G on \mathcal{Q} will imply the existence of such G -actions on any UHF algebra (not necessarily of infinite type). We will eventually reduce to the following case

Lemma 2.5. *Suppose G is a discrete maximally almost periodic group and let $g \in G \setminus \{1\}$. Then for any $(n_l)_{l \in \mathbb{N}}$ there is a sequence $(N_l)_{l \in \mathbb{N}}$ of the same type and a map*

$$\beta[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l}) \xrightarrow{\text{Ad}} \text{Aut } M_{(N_l)_{l \in \mathbb{N}}},$$

such that the image of g , $\beta[g]_g$, is an automorphism with the tracial Rokhlin property.

Proof. Assume that g has infinite order. The case that g has finite order is similar. Suppose α is a tensor product action of G on \mathcal{Q} given by

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q}$$

such that α_g has the tracial Rokhlin property. Since g is a tensor product automorphism with the tracial Rokhlin property on \mathcal{Q} then its restriction to $\bigotimes_{k=n+1}^{\infty} M_k$ is an automorphism and has the tracial Rokhlin property by Lemma 1.25. We can therefore define a function $s : \mathbb{N} \rightarrow \mathbb{N}$ for $n \in \mathbb{N}$ by $s(n)$ is the smallest integer $k \geq n+1$ such that $M_{n+1} \otimes \cdots \otimes M_k$ contains $n' > n$ mutually orthogonal projections $p_1, \dots, p_{n'}$ such that if we write $p = p_1 + \cdots + p_{n'}$, then

- $\alpha_g(p_i) \approx_{1/2^n} p_{i+1}$ for $i \leq n' - 1$,

- $\tau(p) \approx_{1/2^n} 1$.

(If g has finite order we remove the requirement that $n' > n$ and set n' to be the order of g and require $\alpha_g(p_{n'}) \approx_{\epsilon} p_1$.) Now we define the embedding $\beta[g]$ with respect to s . Partition \mathbb{N} into blocks where the i -th block has $s^i(0) - s^{i-1}(0)$ elements and there are no gaps between elements or blocks. We see that if we multiply the numbers together in the i -th block we get, calling the product $S(i)$,

$$S(i) = \frac{s^i(0)!}{s^{i-1}(0)!}.$$

Now we want to partition the integers $(n_l)_{l=1}^{\infty}$ into block that are in bijection with the blocks above but have a much larger size. Let $N(i)$ be the product the numbers in the i -th block here. We want to choose

our partition so that $N(i)$ is large relative to $S(i)$ in the following sense: Upon writing $N(i)$ in quotient remainder form

$$N(i) = Q(i)S(i) + r(i),$$

for unique $Q(i) \in \mathbb{N}$ and $r(i) < S(i)$, we want

$$r(i)/N(i) < 1/2^i.$$

Now embed the unitary groups

$$U(M_{S(i)}) \hookrightarrow U(M_{N(i)})$$

via the block diagonal embedding with $Q(i)$ copies of u and a block of $r(i)$ 1s for each $u \in U(M_{S(i)})$. That is,

$$u \mapsto \text{diag}(u, \dots, u, 1_{r(i)}).$$

Taking the direct product of these embeddings over all $i \in \mathbb{N}$, we get

$$\beta[g] : G \rightarrow \prod_{i=1}^{\infty} U(M_{S(i)}) \hookrightarrow \prod_{i=1}^{\infty} U(M_{N(i)}) \rightarrow \text{Aut} \left(\bigotimes_{i=1}^{\infty} M_{N(i)} \right)$$

We now check that the image of g has the tracial Rokhlin property. Let $\epsilon > 0$, let $\{a_1, \dots, a_k\} \subset A$ be a finite subset and let $n \in \mathbb{N}$. There exists $j \in \mathbb{N}$ and $a_1(j), \dots, a_k(j) \in \bigotimes_{i=1}^{j-1} M_{N(i)}$ such that

- $a_l \approx_{\epsilon} a_l(j)$ for all $l \leq k$
- $j \geq n$
- $1/2^j < \epsilon$.

Now by construction, we can find $n' > j \geq n$ and projections $p_1, p_2, \dots, p_{n'}$ in $M_{S(j)}$ such that

- $\alpha_g(p_i) \approx_{1/2^n} p_{i+1}$ for $i \leq n' - 1$,
- $\tau(p) \approx_{1/2^n} 1$.

We will regard $M_{S(j)}$ as a subalgebra of $M_{N(j)}$ via the embedding

$$M_{S(j)} \hookrightarrow M_{N(j)}$$

via the block diagonal embedding with $Q(j)$ copies of x and one zero block of size $r(j)$, that is

$$x \mapsto \text{diag}(x, \dots, x, 0_{r(j)}).$$

and identify the projections with their image under this embedding. We check that these projections satisfy the properties required by the tracial Rokhlin property.

- Commuting is by construction of placing the projections in a different tensor factor to the finite subset.

$$\begin{aligned} [p_i, a_l] &\approx_{2\epsilon} [p_i, a_l(j)] \\ &= 0. \end{aligned}$$

- We check that the action permutes the projections.

$$\begin{aligned} \beta[g]_g(p_i) &= \beta[g]_g|_{M_{S(j)}}(p_i) \\ &= \alpha_g(p_i) \\ &\approx_{1/2^j} p_{i+1}. \end{aligned}$$

- The only thing that needs to be checked is the trace condition. Let $\tau_{S(j)}$ denote the tracial state on $M_{S(j)}$ with identity $1_{S(j)}$ and $\tau_{N(j)}$ the tracial state on $M_{N(j)}$. Then

$$\begin{aligned} \tau_A(p) &= \tau_{N(j)}(p) \\ &= \tau_{N(j)}(1_{S(j)})\tau_{S(j)}(p) \\ &= \frac{Q(j)S(j)}{N(j)}\tau_{S(j)}(p) \\ &= (1 - \frac{r(j)}{N(j)})\tau_{S(j)}(p) \\ &\approx_\epsilon 1 - \frac{r(j)}{N(j)} \\ &> 1 - 1/2^j \\ &> 1 - \epsilon. \end{aligned}$$

□

We come to our conclusion, which essentially says that any group which can act on a UHF algebra with a tensor product action, does act on every UHF algebra with a strongly outer tensor product action.

Theorem 2.6. *Suppose G is any countable discrete maximally almost periodic group and A is any UHF algebra. Then there exists a strongly outer tensor product action α of G on A . Furthermore, if G is also elementary amenable, then α has the tracial Rokhlin property.*

Proof. Suppose that $A \cong M_{(n_l)_{l \in \mathbb{N}}}$ for some sequence $(n_l)_{l \in \mathbb{N}}$. We can partition this sequence into disjoint subsequences $(n_l(g))_{l \in \mathbb{N}}$ indexed by $g \in G$ whose union is $(n_l)_{l \in \mathbb{N}}$. Since G is a discrete maximally almost periodic group, we have from Lemma 2.5 that for each $g \in G \setminus \{1\}$, there is a sequence $(N_l(g))_{l \in \mathbb{N}}$ of the same type as $(n_l(g))_{l \in \mathbb{N}}$ (and therefore

the union is also the same type) and a map

$$\beta[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l(g)}) \xrightarrow{\text{Ad}} \text{Aut } M_{(N_l(g))_{l \in \mathbb{N}}},$$

such that the image of g , $\beta[g]_g$, is an automorphism with the tracial Rokhlin property. We also assume for notational convenience that $\beta[1] = \text{id}$. Combining these maps into a map to the direct product, we get

$$\beta : G \hookrightarrow \prod_{g \in G} \prod_{l=1}^{\infty} U(M_n) \hookrightarrow \prod_{g \in G} \prod_{l=1}^{\infty} U(M_{N_l(g)}) \xrightarrow{\text{Ad}} \text{Aut} \left(\bigotimes_{g \in G} \bigotimes_{l=1}^{\infty} M_{N_l(g)} \right),$$

where every automorphism in the image will have the tracial Rokhlin property by Lemma 1.25 and hence the action is strongly outer. Re-ordering the factors by Lemma 1.13 we get a conjugate tensor product action (which we will also call β)

$$\beta : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l}) \xrightarrow{\text{Ad}} \text{Aut} \left(\bigotimes_{l=1}^{\infty} M_{N_l} \right),$$

which therefore is also strongly outer. If G is elementary amenable, [19, Theorem 3.7] says that strongly outer actions are equivalent to actions with the tracial Rokhlin property, which completes the proof. \square

It would seem that most of our effort was exerted upon transferring the actions between different UHF algebras and that exhibiting a strongly outer tensor product action on any given UHF algebra of infinite type (as in Proposition 2.3) is quite direct.

This result highlights the robustness of the tracial Rokhlin property. We know this cannot be true for the Rokhlin property because we know that even finite group actions require the UHF algebra to be compatible with the group and essentially of infinite type. We will get a clearer picture of this comparison in the following sections.

3. EXAMPLES OF STRONGLY OUTER TENSOR PRODUCT ACTIONS

We give here a sufficient condition for a tensor product automorphism to be strongly outer in terms of only its trace in $U(M_n)$ for all $n \in \mathbb{N}$. We then use it to exhibit some examples of abelian group actions on \mathcal{Q} .

Let $[u, v]_{U(M_n)} = uvu^*v^*$ denote the commutator inside $U(M_n)$.

Proposition 3.1. *Suppose there are sequences of unitaries such that $\tau([u_n, v_n]_{U(M_n)})$ does not converge to 1. Then $\alpha = \bigotimes \text{Ad } u_n$ and $\beta = \bigotimes \text{Ad } v_n$ are both strongly outer as automorphisms of \mathcal{Q} .*

Proof. Applying Lemma 1.29 with $A_n = M_n$, we check

$$\begin{aligned}
\|\beta(u(n)) - u(n)\|_2^2 &= \tau_n((v_n u_n v_n^* - u_n)^*(v_n u_n v_n^* - u_n)) \\
&= \tau_n(1 - u_n^* v_n u_n v_n^* - v_n u_n^* v_n^* u_n + 1) \\
&= 2 - \tau(u_n^* v_n u_n v_n^*) - \tau(v_n u_n^* v_n^* u_n) \\
&= 2 - \tau(v_n u_n v_n^* u_n^*) - \tau(u_n v_n u_n^* v_n^*) \\
&= 2 - \tau([u_n, v_n]^*) - \tau([u_n, v_n]) \\
&= 2[1 - \operatorname{Re} \tau([u_n, v_n])].
\end{aligned}$$

Note now that if $\tau(w_n)$ does not converge to 1 for $w_n \in U(A)$ then neither does $\operatorname{Re} \tau(w_n)$. Hence we see that β is strongly outer. Now $\tau([u_n, v_n]) = \tau([v_n, u_n^*])$ implies that α^{-1} is strongly outer and hence α is strongly outer. \square

Lemma 3.2. *There is a strongly outer action*

$$\mathbb{R}/\mathbb{Z} \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\operatorname{Ad}} \operatorname{Aut} \mathcal{Q},$$

Proof. Define a homomorphism $\mathbb{R} \rightarrow U(M_n)$ by

$$v_n(r) = \operatorname{diag}(e^{2\pi i l r})_{l=1}^n,$$

whose kernel is \mathbb{Z} , so we get an injective map

$$\mathbb{R}/\mathbb{Z} \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\operatorname{Ad}} \operatorname{Aut} \mathcal{Q}.$$

We use the condition in Proposition 3.1 to check that it is strongly outer. Let u_n be the permutation matrix corresponding to the cycle $(123 \dots n)$. Then

$$\begin{aligned}
\tau([u_n, v_n(r)]) &= \tau(\operatorname{Ad} u_n(v_n(r))v_n(r)^*) \\
&= \tau(\operatorname{diag}(e^{2\pi i n r}, e^{-2\pi i r}, \dots, e^{2\pi i(n-1)r}) \operatorname{diag}(e^{-2\pi i l r})_{l=1}^n) \\
&= \tau(\operatorname{diag}(e^{2(n-1)\pi i r}, e^{-2\pi i r}, e^{-2\pi i r}, \dots, e^{-2\pi i r})) \\
&= \frac{1}{n}(e^{2(n-1)\pi i r} + (n-1)e^{-2\pi i r}) \\
&= e^{-2\pi i r}(n^{-1}e^{-2n\pi i r} + n^{-1}(n-1)) \\
&\rightarrow e^{-2\pi i r}.
\end{aligned}$$

\square

Lemma 3.3. *There is a strongly outer action*

$$\mathbb{R} \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\operatorname{Ad}} \operatorname{Aut} \mathcal{Q},$$

Proof. Let θ be an irrational number and let

$$\theta_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \theta & \text{if } n \text{ is even.} \end{cases}$$

For each $r \in \mathbb{R}$ and $n \in \mathbb{N}$, define a homomorphism $\mathbb{R} \rightarrow U(M_n)$ by

$$v_n(r) = \text{diag}(e^{2\pi\theta_n i l r})_{l=1}^n,$$

We again use the condition in Proposition 3.1 to check that it is strongly outer. Let u_n be the permutation matrix corresponding to the cycle $(123 \dots n)$. Then

$$\begin{aligned} \tau([u_n, v_n(r)]) &= \tau(\text{Ad } u_n(v_n(r))v_n(r)^*) \\ &= \tau(\text{diag}(e^{2\pi\theta_n i n r}, e^{2\pi\theta_n i r}, \dots, e^{2\pi\theta_n i(n-1)r}) \text{diag}(e^{-2\pi\theta_n i l r})_{l=1}^n) \\ &= \tau(\text{diag}(e^{2(n-1)\pi\theta_n i r}, e^{-2\pi\theta_n i r}, e^{-2\pi\theta_n i r}, \dots, e^{-2\pi\theta_n i r})) \\ &= \frac{1}{n}(e^{2(n-1)\pi\theta_n i r} + (n-1)e^{-2\pi\theta_n i r}) \\ &= e^{-2\pi\theta_n i r}(n^{-1}e^{-2n\pi i r} + n^{-1}(n-1)). \end{aligned}$$

Now two of the limit points of this, corresponding to odd and even n , are $e^{-2\pi\theta i r}$ and $e^{-2\pi i r}$. If these are both equal to 1 the r must be both irrational in the first case and irrational in the second, which is impossible unless $r = 0$. \square

Corollary 3.4. *There exist strongly outer tensor product actions of \mathbb{Q} and \mathbb{Q}/\mathbb{Z} on \mathcal{Q} .*

Proof. For \mathbb{Q} , restrict the homomorphism in Lemma 3.3 from \mathbb{R} to \mathbb{Q} and for \mathbb{Q}/\mathbb{Z} restrict the map from Lemma 3.2. \square

The above two lemmas represent the hardest case of abelian groups in two senses. The first is that these groups are not residually finite and the second is that showing they can act with strongly outer tensor product actions actually implies that every countable abelian group can also using the next lemma.

Lemma 3.5. *If for each $j \in \mathbb{N}$, G_j has a strongly outer tensor product action of the form*

$$\alpha_j : G_j \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q},$$

then there exists a strongly outer tensor product action α of the infinite direct sum $G = \bigoplus_{j \in \mathbb{N}} G_j$ on \mathcal{Q} .

Proof. We will try to fit each of the direct summands into mutually disjoint subsequences of factors. For this it suffices to show that there is an appropriate embedding

$$\left(\prod_{n=1}^{\infty} U(M_n)\right) \oplus \left(\prod_{n=1}^{\infty} U(M_n)\right) \hookrightarrow \prod_{n=1}^{\infty} U(M_n)$$

First we see that there is an obvious embedding

$$\prod_{n=1}^{\infty} U(M_{n^2}) \hookrightarrow \prod_{n=1}^{\infty} U(M_n),$$

which when combined with the obvious embedding

$$U(M_n) \times U(M_n) \hookrightarrow U(M_{n^2})$$

gives the required embedding. This embedding is appropriate because when we restrict to G_j , the action is of the form $\alpha_j \otimes \text{id}$, which is strongly outer. □

We have enough now to get a strongly outer tensor product action of any countable discrete abelian group.

Corollary 3.6. *Every countable discrete abelian group G has a strongly outer tensor product action of the form*

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\text{Ad}} \text{Aut } \mathcal{Q},$$

Proof. Combining Lemmas 3.3 and 3.5 we see that there exists a strongly outer tensor product action of

$$\bigoplus_{n=1}^{\infty} \mathbb{Q} \oplus \bigoplus_{n=1}^{\infty} \mathbb{Q}/\mathbb{Z}$$

on \mathcal{Q} . We know from Proposition 1.9 that every countable abelian group appears as a subgroup of this group. □

4. TENSOR PRODUCT ACTIONS WITH THE ROKHLIN PROPERTY

We begin this section by recalling Proposition 1.24, which says that actions of a finite group G can be found on $A \otimes M_{|G|^\infty}$. We first generalise this to include groups that are certain extensions by finite groups and record it as Theorem 4.3. We then use this along with our examples from Section 3 to show that almost abelian groups can act on \mathcal{Q} with the pointwise Rokhlin property. In particular this result will be

independent of Section 2. We then conclude with some remarks about why this is about as good as one can hope for.

Proposition 4.1. *Let G be a discrete group with a normal subgroup N and let $q : G \rightarrow G/N$ be the quotient map. Suppose A and B are unital nuclear C^* -algebras, α is an action of G on A such that $\alpha|_N$ has the pointwise Rokhlin property, and β is an action of G/N on B that has the pointwise Rokhlin property. Then the action γ of G on $A \otimes B$, defined by $\gamma = \alpha \otimes (\beta \circ q)$ has the pointwise Rokhlin property.*

Proof. Note that if $g \in N$, then α_g is an automorphism with the pointwise Rokhlin property and if $g \notin N$, then $(\beta \circ q)_g$ is an automorphism with the pointwise Rokhlin property. We will now try to apply Lemma 1.25 which involves considering the order of γ_g .

- Suppose γ_g has infinite order and $(\beta \circ q)_g$ has infinite order, then $g \notin N$ meaning that $(\beta \circ q)_g$ has the pointwise Rokhlin property and we are done by applying Lemma 1.25 with the roles of A and B reversed.
- If $(\beta \circ q)_g$ has finite order k and α_g has infinite order, then $g^k \in N$ and so $\alpha_{g^k} = \alpha_{g^k}$ has the pointwise Rokhlin property and $(\beta \circ q)_g$ has the pointwise Rokhlin property. This case is the content of Lemma 1.25.
- If γ_g has finite order, then the orders of α_g and $(\beta \circ q)_g$ are also finite. Let k be the order of $(\beta \circ q)_g$. Then once again, $g^k \in N$ and α_{g^k} have the pointwise Rokhlin property, so Lemma 1.25 applies.

□

Lemma 4.2. *Suppose G is a countable discrete group, H is a subgroup of G with finite index k and let*

$$N = \bigcap_{g \in G} gHg^{-1}.$$

Also suppose that A is a UHF algebra and there is a tensor product action

$$\alpha_H : H \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \rightarrow \text{Aut } A,$$

of H on A with the pointwise Rokhlin property, then there is an extension of α to a tensor product action

$$\alpha_H^G : G \rightarrow \prod_{l=1}^{\infty} U(M_{n_l} \otimes M_k) \rightarrow \text{Aut} \left(M_{(n_l)_{l \in \mathbb{N}}} \otimes M_k \right),$$

of G on $A \otimes M_{k^\infty}$ whose restriction to N has the pointwise Rokhlin property.

Proof. For each homomorphism $H \rightarrow U(M_{n_l})$ coming from α_H apply Lemma 1.7 to get an induced homomorphism $G \rightarrow U(M_{n_l} \otimes M_k)$ such that when we put all of the maps together we get a map

$$\alpha_H^G : G \rightarrow \prod_{n=1}^{\infty} U(M_n \otimes M_k) \rightarrow \text{Aut} \left(\bigotimes_{n=1}^{\infty} (M_{n_l} \otimes M_k) \right)$$

such that $\alpha_H^G|_N$ is conjugate to the action $\alpha_H|_N \otimes \text{id}_{M_{k^\infty}}$, which has the pointwise Rokhlin property by Lemma 1.25. \square

Theorem 4.3. *Suppose G is a countable discrete group with a subgroup H of finite index and normal subgroup of index k given by*

$$N = \bigcap_{g \in G} gHg^{-1}.$$

Suppose also that A is any UHF algebra. If H has a tensor product action on A with the (pointwise) Rokhlin property, then there is a tensor product action of G on $A \otimes M_{k^\infty}$ with the (pointwise) Rokhlin property.

Proof. Let q be the quotient map by N . By assumption, there exists a tensor product action

$$\alpha_H : H \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \rightarrow \text{Aut } A,$$

of H on A has the pointwise Rokhlin property. Let k' be the index of H which we note divides k . We then have by Proposition 4.2 an action

$$\alpha_H^G : G \rightarrow \prod_{l=1}^{\infty} U(M_{n_l} \otimes M_{k'}) \rightarrow \text{Aut } M_{(n_l k')_{l \in \mathbb{N}}},$$

of G on $A \otimes M_{k^\infty}$ whose restriction to N has the pointwise Rokhlin property. Now since G/N is a finite group of size k then we have by Proposition 1.24 that there is a tensor product action

$$\beta : G/H \rightarrow \prod_{l=1}^{\infty} U(M_k) \rightarrow \text{Aut } M_{k^\infty}$$

of G/N on M_{k^∞} with the pointwise Rokhlin property. Combine these in the way specified by Proposition 4.1 to get an action $\alpha_H^G \otimes (\beta \circ q)$ of G on $M_{(n_l k')_{l \in \mathbb{N}}} \otimes M_{k^\infty}$ with the pointwise Rokhlin property and that we easily see is of the form

$$G \hookrightarrow \left(\prod_{l=1}^{\infty} U(M_{n_l k}) \right) \times \left(\prod_{l=1}^{\infty} U(M_k) \right) \rightarrow \text{Aut}(M_{(n_l k')_{l \in \mathbb{N}}} \otimes M_{k^\infty}),$$

Reordering the factors we get a conjugate action (keeping the same name) of the form

$$\alpha_H^G \otimes (\beta \circ q) : G \hookrightarrow \prod_{n=1}^{\infty} U(M_{n_l k' k}) \rightarrow \text{Aut}(M_{(n_l k' k)_{l \in \mathbb{N}}}).$$

Since k' divides k we have that $M_{(n_l k' k)_{l \in \mathbb{N}}} \cong A \otimes M_{k^\infty}$ and we are done. \square

We notice that when H is trivial, G is a finite group and thus we have generalised Proposition 1.24. We now look to the UHF algebra \mathcal{Q} , the only algebra that can possibly have an existence theorem for the pointwise Rokhlin property like that of Theorem 2.6.

Lemma 4.4. *Suppose G is a discrete abelian group and let $g \in G$ with finite order k . Suppose α is a tensor product action of G on some UHF algebra A such that α_g has the tracial Rokhlin property. Then there is a map*

$$\alpha[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_k) \hookrightarrow \text{Aut } M_{k^\infty}$$

such that the image of g , $\alpha[g]_g$, is an automorphism with the strict Rokhlin property. We also assume for notational convenience that $\alpha[1] = \text{id}$.

Proof. Since α_g has the tracial Rokhlin property we can regroup the factors of A so that we can write

$$A \cong M_{(n_l)_{l \in \mathbb{N}}}$$

where for each $l \in \mathbb{N}$ there are k mutually orthogonal projections $p_1^{(l)}, \dots, p_{k-1}^{(l)} \in M_{n_l}$ such that

- $\alpha_g(p_i^{(l)}) \approx_{\frac{1}{l^2}} p_{i+1}^{(l)}$
- $p_k^{(l)} = p_1^{(l)}$.

Let g_{n_l} be the unitary representing g on M_{n_l} so that $\alpha_g|_{M_{n_l}} = \text{Ad } g_{n_l}$. We can decompose \mathbb{C}^{n_l} into the eigenspaces of g_{n_l} as follows

$$\mathbb{C}^{n_l} = \bigoplus_{j=0}^{k-1} V_j,$$

where V_i is the $e^{2\pi i j/k}$ -eigenspace of g_{n_l} . By taking a subprojection if necessary we can assume that p_1 has rank one. Let v be a unit vector that spans the rank of p_1 , then we write for unique $v_j \in V_j$

$$v = \sum_{j=1}^{k-1} v_j$$

Now restrict g_{n_l} and p_1 to the subspace spanned by $\{v_i \mid i \leq k-1\}$ which both preserve. Hence we also have that $\alpha_g^j(p_1)$ preserve this subspace for all $j \leq k-1$. These are approximately mutually orthogonal projections (for large enough l) and therefore by functional calculus can be replaced by exactly mutually orthogonal projections. Thus $\{v_i \mid i \leq k-1\}$ is a linearly independent set, in particular with every vector non-zero. Hence there are k distinct eigenvalues in the spectrum of g_{n_l} . Now since G is abelian, we can simultaneously diagonalise the unitaries in the image of G in $U(M_{n_l})$. So for each j there is an $e^{2\pi i j/k}$ -eigenvector preserved by all of G . Restricting to the subspace spanned by these vectors we get a map

$$U(M_{n_l}) \rightarrow U(M_k)$$

where the image of g_{n_l} in some basis acts as a cyclic permutation of the diagonal projections $e_{j,j}$ and hence has the strict Rokhlin property. \square

Theorem 4.5. *Let \mathcal{Q} be the universal UHF algebra and let G be any countable discrete almost abelian group. Then there exists a tensor product action*

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \rightarrow \text{Aut } \mathcal{Q}$$

of G on \mathcal{Q} with the pointwise Rokhlin property.

Proof. By Theorem 4.3 it suffices to assume that G is abelian. In this case combining Corollary 3.6 with Lemma 4.4 we have for each $g \in G$ of finite order $k(g)$, a map

$$\alpha[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{k(g)}) \hookrightarrow \text{Aut } M_{k(g)}^{\infty}$$

such that the image of g , $\alpha[g]_g$, is an automorphism with the strict Rokhlin property. For $g \in G$ with infinite order, let $\alpha[g]$ be any strongly outer tensor product action given by Lemma 3.6 since g will have the Rokhlin property in this case because we have from Kishimoto [14, Theorem 1.4] every automorphism with the tracial Rokhlin property on \mathcal{Q} is outer conjugate and in particular outer conjugate to the one from Proposition 1.23 which has the Rokhlin property on $M_{2^\infty} \otimes \mathcal{Q}$. We can assume for all $g \in G$ that our maps are of the form

$$\alpha[g] : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \hookrightarrow \text{Aut } \mathcal{Q}.$$

Then combining these maps we get a map to the direct product

$$\alpha : G \hookrightarrow \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_n) \hookrightarrow \text{Aut} \left(\bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_n \right)$$

where every $g \in G$ acts via an automorphism with the strict Rokhlin property. Hence α has the pointwise Rokhlin property. We now apply Lemma 1.13 to get a tensor product action conjugate to β of the form required. \square

Notice that since we made use of the example in Section 3 to provide the strongly outer actions of abelian groups, the above proof is logically independent of the results in Section 2.

We look at an example which was studied in Matui-Sato [19] as the first classification result for a group that was not finite nor abelian. It is however almost abelian.

Example 4.6 (Klein bottle group). The Klein bottle group has presentation

$$\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b \mid bab^{-1} = a^{-1} \rangle.$$

We will note annotate the action since there is only one non-trivial action of \mathbb{Z} on \mathbb{Z} . It was shown in Matui-Sato [19, Theorem 7.9] that every strongly outer action of $\mathbb{Z} \rtimes \mathbb{Z}$ on a UHF algebra A that absorbs M_{2^∞} is equivalent at least in the sense that the crossed products $A \rtimes (\mathbb{Z} \rtimes \mathbb{Z})$ are all the same (their equivalence is stronger) and the same is true on a UHF algebra where M_2 does not appear in any decomposition. They also showed the crossed product $\mathcal{Q} \rtimes (\mathbb{Z} \rtimes \mathbb{Z})$ is unital simple separable nuclear has tracial rank zero and satisfies the UCT. In both cases the K_0 -groups are given by

$$K_0(A \rtimes (\mathbb{Z} \rtimes \mathbb{Z})) = K_0(A),$$

while in the first case

$$K_1(A \rtimes (\mathbb{Z} \rtimes \mathbb{Z})) = K_0(A)$$

but in the second

$$K_1(A \rtimes (\mathbb{Z} \rtimes \mathbb{Z})) = K_0(A) \oplus \mathbb{Z}/2\mathbb{Z}.$$

We will present here a model action of $\mathbb{Z} \rtimes \mathbb{Z}$ on M_{2^∞} with the pointwise Rokhlin property using the principles of Theorem 4.5. Notice that $\mathbb{Z} \rtimes \mathbb{Z}$ is an almost abelian group with subgroup N generated by a and b^2 isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This subgroup clearly has index 2 and is therefore normal. We have a \mathbb{Z} action on M_{2^∞} given by Proposition 1.23. Duplicating this map we get a map $\mathbb{Z} \oplus \mathbb{Z} \rightarrow U(M_{2^l}) \times U(M_{2^l})$ embedding this into $U(M_{2^{l+1}})$ we get a tensor product action of $\mathbb{Z} \oplus \mathbb{Z}$ on M_{2^∞} with the pointwise Rokhlin property. Now extend to $\mathbb{Z} \rtimes \mathbb{Z} \rightarrow U(M_{2^{l+1}} \otimes M_2)$ by induction as in Lemma 1.7. Since N is normal we see that this will be enough to get the pointwise Rokhlin property upon taking a direct product. Let $g \notin N$ we see that there is a basis for $\mathbb{C}^{2^{l+1}} \otimes \mathbb{C}^2$ for which g acts for some $n \in N$ as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes n.$$

Putting this together as an infinite tensor product we have by Lemma 1.25 that g acts with the Rokhlin property. Hence we have a model action of $\mathbb{Z} \rtimes \mathbb{Z}$ on M_{2^∞} with the pointwise Rokhlin property.

5. THE CROSSED PRODUCTS $A \rtimes_\alpha G$

Let A be a unital C^* -algebra, let G be a countable discrete group and let α be an action of G on A . It is clear that $A \rtimes_\alpha G$ is unital and separable. We will also assume that A is a UHF algebra and note the advantages for determining $A \rtimes_\alpha G$ when α is a tensor product action. Nonetheless this is not enough to get everything that one would desire. For example, we know that with only this we cannot guarantee that $A \rtimes_\alpha G$ has a unique tracial state or has real rank zero (see for example [14, Theorem 1.3]). However, these missing properties are exactly why we looked for the tracial Rokhlin property.

We combine the advantages of the tracial Rokhlin property with being a tensor product action in the following theorem.

Theorem 5.1. *Suppose G is a countable discrete maximally almost periodic amenable group, A is any UHF algebra and α is a tensor product action of G on A with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ is unital simple separable nuclear with tracial rank zero and satisfies the*

Universal Coefficient Theorem. Moreover, $A \rtimes_\alpha G$ has a unique tracial state.

Proof. By a result of W. Winter [27, Theorem 2.1], to prove $A \rtimes_\alpha G$ has tracial rank zero, it suffices to show $A \rtimes_\alpha G$ is \mathcal{Z} -stable with real rank zero and finite decomposition rank. We have real rank zero and \mathcal{Z} -stability from Proposition 1.32 (iii). So it suffices to show that $A \rtimes_\alpha G$ has finite decomposition rank. Now using Matui Sato [20, Corollary 1.2], it suffices to show that $A \rtimes_\alpha G$ is simple nuclear quasidiagonal with a unique tracial state. Simplicity and nuclearity come from Proposition 1.32 (i), while quasidiagonal follows from Corollary 1.18. Having a unique tracial state comes from Proposition 1.32 (ii). The Universal Coefficient Theorem is from Corollary 1.18. \square

Corollary 5.2. *Suppose G is a countable discrete maximally almost periodic elementary amenable group, A is any UHF algebra and α is a strongly outer tensor product action of G on A . Then $A \rtimes_\alpha G$ is unital simple separable nuclear with tracial rank zero and satisfies the Universal Coefficient Theorem. Moreover, $A \rtimes_\alpha G$ has a unique tracial state. Furthermore, if G is almost abelian then $A \rtimes_\alpha G$ is locally type I.*

Proof. When G is elementary amenable, the tracial Rokhlin property is equivalent to strong outeriness by Matui-Sato's theorem, reproduced here as Theorem 1.30. Hence Theorem 5.1 applies. \square

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